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1993 J. Phys. A: Math. Gen. 26 3045

(http://iopscience.iop.org/0305-4470/26/13/009)

J. Phys. A: Math. Gen. 26 (1993) 3045–3062. Printed in the UK

# The correlation length of the Potts model at the first-order transition point

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Received 23 July 1992, in final form 22 March 1993

**Abstract.** We consider the $Q$-state two-dimensional Potts model for $Q > 4$, i.e. in the first-order phase transition regime. Following a scheme given by P Martin, we prove an identity between the spectra of the transfer matrices of the Potts model and the transfer matrices connecting the diagonals of a 6-vertex model. By using a Bethe ansatz for the latter, we obtain an exact expression for the correlation length of the Potts model at the transition point.

## 1. Introduction

The square lattice 2D Potts model with $Q > 4$ is widely used as a testing ground for ideas about first-order phase transitions; for instance, methods for determining the order of a phase transition by numerical studies of finite systems have often been tested against the 2D Potts model. It is therefore useful to have at our disposal exact results about this model, in particular the correlation length, which allow us to determine the finite-size effects.

It is commonly admitted that there is an equivalence between the Potts model and the diagonal 6-vertex model which allows the calculation of free energy. In order to determine the correlation length we must find the next largest eigenvalue of the Potts model and thus we must study more precisely the equivalence between the spectra of the transfer matrices of both models.

In this paper, following a scheme given by Martin [1] we prove the identity of these spectra (section 2). Then, at the transition temperature, we solve the homogeneous diagonal 6-vertex model, using a Bethe ansatz given by Owczarek and Baxter [2] (section 3).

Through a hyperbolic parametrization we find analytic expressions for the free energy and the correlation length (4.44) and (4.46). The behaviour of the latter for $Q \longrightarrow 4$ is derived in the appendix.

### 1.1. The Potts model

The $Q$-state Potts model [3,4] is a generalization of the Ising spin model. The spin variables $\sigma$, associated with the sites of a 2D square lattice, can take $Q$ different values. The Hamiltonian of the model is, in the isotropic case considered in this paper,

$$H = -J \sum_{(i,j)} \delta(\sigma_i, \sigma_j) \tag{1.1}$$

where the summation is over all edges $(i, j)$ of the graph.

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Consider two successive rows. Let $\phi = \{\sigma_1, \ldots, \sigma_N\}$ be the spins in the lower row and $\phi' = \{\sigma'_1, \ldots, \sigma'_N\}$ the spins in the upper row. The Boltzmann weights corresponding to the addition of $\phi'$ is described by the matrix $VW$ where

$$\begin{aligned} V_{\sigma,\sigma'} &= \exp\left(K \sum_{j=1}^{N-1} \delta(\sigma_j, \sigma_{j+1})\right) \prod_{j=1}^{N} \delta(\sigma_j, \sigma'_j) && \text{describes the horizontal interactions} \\ W_{\sigma,\sigma'} &= \exp\left(K \sum_{j=1}^{N} \delta(\sigma_j, \sigma'_j)\right) && \text{describes the vertical interactions} \end{aligned} \tag{1.2}$$

(with $K = J/k_{\rm B}T$). We can write the partition function of a lattice with $M$ rows and $N$ columns with free boundaries as

$$Z_N = \xi^\dagger (VW)(VW)\cdots(VW)(VW)V\xi \tag{1.3}$$

containing $M$ factors $V$ and $M-1$ factors $W$; $\xi$ is a $Q^N$-dimensional vector whose entries are all unity.

As shown by Baxter [5], the correlation length along a column is

$$\xi = \frac{1}{\ln(\Lambda_{\max}/\Lambda_2)} \tag{1.4}$$

where $\Lambda_{\max}$ is the largest eigenvalue of the transfer matrix and $\Lambda_2$ is the next largest eigenvalue not degenerate with $\Lambda_{\max}$ in the thermodynamic limit.

## 2. Spectra of the Potts model and the 6-vertex model

In this section, we shall prove the identity of the transfer matrix spectra for the Potts model and for the particular 6-vertex model under consideration.

### 2.1. Temperley–Lieb algebra representations

We consider the *Temperley–Lieb algebra*, defined by $2N$ generators $U_i$ which verify

$$U_i^2 = Q^{1/2} U_i \qquad \forall i \tag{2.1a}$$

$$U_i U_j = U_j U_i \qquad \text{with } |i-j| \geqslant 2,\ \forall i, j \tag{2.1b}$$

$$U_i U_{i\pm1} U_i = U_i \qquad \forall i. \tag{2.1c}$$

This algebra is denoted $T_{2N}(Q)$. We then introduce the two particular following representations (see Baxter [5] ch 12).

(i) *The Potts representation.* This acts on $(\sigma_1, \ldots, \sigma_N)$, with $\sigma_i$ taking its values in a set of $Q$ values. The representation is a set of $Q^N \times Q^N$ matrices defined by

$$\begin{aligned} (U_{2i})_{\sigma,\sigma'} &= Q^{1/2} \delta(\sigma_i, \sigma_{i+1}) \prod_{1\leqslant j\leqslant N} \delta(\sigma_j, \sigma'_j) \\ (U_{2i-1})_{\sigma,\sigma'} &= Q^{-1/2} \prod_{1\leqslant j\neq i\leqslant N} \delta(\sigma_j, \sigma'_j) \end{aligned} \tag{2.2}$$

It is readily observed that

$$V = \prod_{j=1}^{N-1} (\mathcal{J} + x\, U_{2j}) \tag{2.3a}$$

$$W = (\sqrt{Q}\, x)^N \prod_{j=1}^{N} \left(\mathcal{J} + \frac{1}{x} U_{2j-1}\right) \tag{2.3b}$$

with $x = \sqrt{Q}(\exp K - 1)$ and $\mathcal{J}$ is the identity operator.

(ii) *The 6-vertex representation.* This acts on $(\sigma_1, \dots, \sigma_{2N})$, with $\sigma_i$ taking its values in $\{-1, +1\}$. The representation is a set of $4^N \times 4^N$ matrices defined by

$$(U_i)_{\sigma,\sigma'} = \delta(\sigma_1, \sigma_1') \cdots \delta(\sigma_{i-1}, \sigma_{i-1}') h(\sigma_i, \sigma_{i+1}) h(\sigma_i', \sigma_{i+1}') \delta(\sigma_{i+2}, \sigma_{i+2}') \cdots \delta(\sigma_{2N}, \sigma_{2N}') \tag{2.4}$$

where

$$h(+,+) = h(-,-) = 0 \qquad h(+,-) = \mathrm{e}^{-\lambda/2} \qquad h(-,+) = \mathrm{e}^{\lambda/2}$$

with $2\cosh(\lambda) = \sqrt{Q}$.

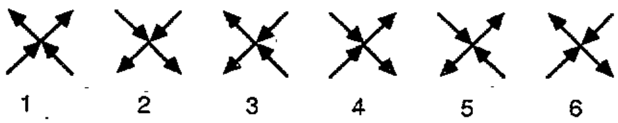
One can regard $(\sigma_1, \dots, \sigma_{2N})$ as representing a row of near vertical arrows:

| | |
|---|---|
| $\sigma_j = +$ | if the arrow in column $j$ points up |
| $\sigma_j = -$ | if it points down. |

We now introduce the operators $V$ and $W$, defined by equations (2.3). One can verify that $V$ is, in fact, the transfer matrix $T_1$ of a row of sites of type 1 in a 6-vertex inhomogeneous model generated by the diagonals with Boltzmann weights:

$$\omega_1 = 1 \qquad \omega_2 = 1 \qquad \omega_3 = x \qquad \omega_4 = x \qquad \omega_5 = 1 + x\mathrm{e}^{\lambda} \qquad \omega_6 = 1 + x\mathrm{e}^{-\lambda}.$$

They are associated with the six arrow arrangements respecting the ice rule.




Likewise $q^{-N/2}W$ is the transfer matrix $T_2$ for a row of sites of type 2 in the same inhomogeneous 6-vertex model, with weights

$$\omega_1 = x \qquad \omega_2 = x \qquad \omega_3 = 1 \qquad \omega_4 = 1 \qquad \omega_5 = x + \mathrm{e}^{\lambda} \qquad \omega_6 = x + \mathrm{e}^{-\lambda}$$

(see Baxter [5], pp 334–7).

Figure 1 illustrates the correspondence between the two models. In what follows we only consider the transition point $x = 1$. Furthermore, since $\omega_5$ and $\omega_6$ always appear the same number of times (because of the ice rule), we replace $\omega_5$ and $\omega_6$ by $\sqrt{\omega_5 - \omega_6}$. We thus end up with a homogeneous diagonal 6-vertex model with weights

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1 \qquad \omega_5 = \omega_6 = c = \sqrt{2 + \sqrt{Q}}. \tag{2.5}$$

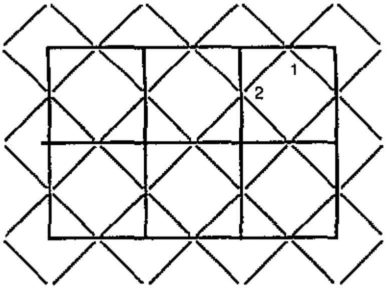
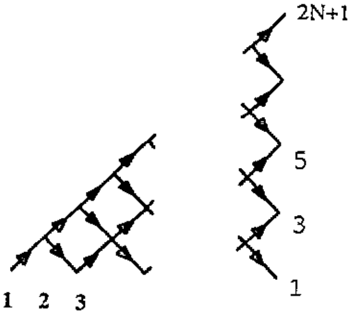



**Figure 1.** The square lattice of the Potts model (in bold) and the associated 6-vertex lattice. The two classes of site of the 6-vertex model are indicated (they correspond to the vertical and horizontal edges of the square lattice). In this example, $N = 4$ and $M = 3$.

### 2.2. *The regular representation of the Temperley–Lieb algebra*

Before studying the completeness of the two representations described above, we are going to describe a representation of this algebra which contains all irreducible components. This is the regular representation, defined by letting the algebra act on itself. We are then led to study the action of $T_{2N}(Q)$ on the products of $U_i$. As shown by Jones [6], we can put these products in a 'reduct-form'. *It is interesting to consider a representation which is related to these reduct-form of the products and to index them like the walks written on the so-called Bratelli diagram.*




It is obvious that we can define a walk on this diagram by a sequence $\{s\}$ of $2N+1$ integers which verify

$$s_0 = 1 \qquad s_i \geqslant 1 \qquad |s_i - s_{i+1}| = 1 \qquad \forall i.$$

Then we call $S(2N)$ the set of sequences of length $2N + 1$, i.e. the set of walks on the diagram; $S(2N, m)$ the set of walks which end up at $m$; and $P(2N)$ and $P(2N, m)$ the sets of pairs of elements of $S(2N)$ and $S(2N, m)$ respectively.

*In order to find all irreducible components of this algebra, we need to exhibit a set of* central idempotent operators (then the matrices will break into diagonal blocks) and then to show that the entire algebra can be written as a direct sum of ideals generated by these idempotents.

We first identify a set of idempotents.

Let $I_m \in T_{2N}(Q)$ with $m = 0, 1, \ldots, 2N$ be defined by

$$I_0 = \mathcal{J} \qquad \text{and} \qquad I_{m+1} = I_m(\mathcal{J} - k_{m+2}U_{m+1})I_m \tag{2.6}$$

where $k_n$ is chosen such that $\sqrt{Q} - k_n = 1/(k_{n+1})$, i.e.

$$k_n = \frac{\sinh((n-1)\pi/r)}{\sinh((n)\pi/r)} \qquad \text{with } r \text{ such that } Q = 4\cosh^2\left(\frac{\pi}{r}\right). \tag{2.7}$$

We can easily prove by recursion that

$$(I_b U_{b+1})^2 = k_{b+2}^{-1} I_b U_{b+1} \qquad I_b^2 = I_b \qquad U_i I_b = I_b U_i = 0 \qquad \forall i \leqslant b. \tag{2.8}$$

The set of operators $E_m$ defined by

$$E_m = I_{m-2} \qquad \text{for } m = 2, \ldots, 2N+2 \qquad \text{and} \qquad E_1 = I_d \tag{2.9}$$

is a set of idempotents. Let us study the ideals generated by these idempotents.

We now introduce the representation of $T_{2N}(Q)$ which is associated with the Bratelli diagrams, find a free family of operators, and deduce the independence between all ideals.

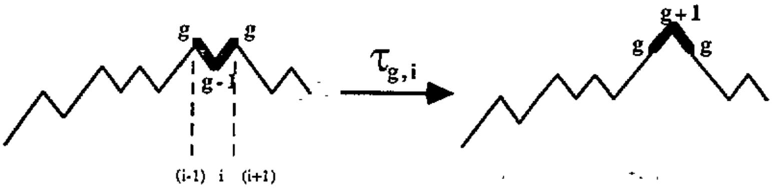
The representation of the algebra $T_{2N}(Q)$ acts in a space which has the walks on the Bratelli diagram (i.e. the $\{s\} \in S(2N)$) as a basis. The matrices are linear combinations of the matrices $(s, t)$ which transform a basis vector in another one with the definition

$$\{s\} \xrightarrow{(s,t)} \{t\} \qquad \text{and} \qquad \{s'\} \xrightarrow{(s,t)} 0.$$

In order to show that the above definitions define a representation of $T_{2N}(Q)$ we must associate with each matrix an operator of $T_{2N}(Q)$ and verify that the elementary product of matrices in this representation corresponds to the associated product of operators.

Let us first associate an operator with each element $(s, t)$ of $P(2N)$:

Considering the elementary 'action' $\tau_{g,i}$ on a sequence $\{s\}$ which transform $\{s\}$ into $\{s'\}$ with $\{s'\}$ differing from $\{s\}$ only at the $i$th position:




(2.10)

We easily observe that we can construct each $\{s\} \in S(2N, m)$ from another walk $\{s'\}$ below $\{s\}$ by iterating the 'actions' $\tau_{g,i}$. In particular, for every $\{s\} \in S(2N, m)$ we can begin the construction with the walk $e_m$ defined by


$$= \quad e_m \tag{2.11}$$

We use this construction to define a correspondence between pairs of walks on the diagram and operators of $T_{2N}(Q)$. We introduce the notation $p$ for $\frac{1}{2}(2N + 1 - m)$ and $E_m^{(2p)}$ is then obtained from $E_m$ by replacing each $U_i$ by $U_{i+2p}$ in the writing of $E_m$ in terms of the generators of $T_{2N}(Q)$.

$$\begin{array}{lcl}
\text{we begin with } (e_m, e_m) & \leftrightarrow & \text{we begin with } \displaystyle\prod_{1 \leqslant i \leqslant p} U_{2i-1} E_m^{(2p)} \\
\Big\downarrow \boxed{\begin{array}{c}\text{iteration of} \\ \tau_{g,i} \text{ at left}\end{array}} & \leftrightarrow & \text{multiplication by } \sqrt{k_g k_{g+1}} \left( \mathcal{J} - \dfrac{U_i}{k_g} \right) \text{ at left} \\
(s, e_m) & & \\
\Big\downarrow \boxed{\begin{array}{c}\text{iteration of} \\ \tau_{g,i} \text{ at right}\end{array}} & \leftrightarrow & \text{multiplication by } \sqrt{k_g k_{g+1}} \left( \mathcal{J} - \dfrac{U_i}{k_g} \right) \text{ at right.} \\
(s, t) \in P(2N, m) & &
\end{array} \tag{2.12}$$

Following this scheme, $(s, t)$ is associated with an operator of $T_{2N}(Q)$ called $\mathcal{T}(s, t)$.

We can justify the consistency of our construction, since the definition of $(s, t)$ is unique, i.e. it is independent of the choice of order in which the identities may be applied in moving from the initially defined operator $(e_m, e_m)$ to $(s, t)$. This derives from the relation (2.1*c*).

With our definitions, we can verify the relation

$$\mathcal{T}(u, s) \cdot \mathcal{T}(t, v) = \delta_{st}\mathcal{T}(u, v) \tag{2.13}$$

which means that the algebra of matrices $(u, s) \in P(2N)$ is indeed a representation of the algebra $T_{2N}(Q)$ because the laws of product are conserved through $\mathcal{T}$. ((2.13) is obtained by a recursion on the pair $(s, t)$, with respect to the lexicographical order in $P(2N)$).

From (2.13) we can also deduce trivially that $\{\mathcal{T}(s, t)/(s, t) \in P(2N)\}$ is a free family of operators. This states the independence between the ideals generated by the idempotents.

We must now prove that the whole algebra can be covered by these operators. To this aim, we associate new operators of $T_{2N}(Q)$ with each element of $P(2N)$:

$$\begin{array}{ccl}
\text{we begin with } (e_m, e_m) & \leftrightarrow & \text{we begin with } \displaystyle\prod_{1\leqslant i\leqslant p} U_{2i-1} \\
\Big\downarrow \boxed{\begin{array}{c}\text{iteration of}\\ \tau_{g,l} \text{ at left}\end{array}} & \leftrightarrow & \text{multiplication by } U_i \text{ at left} \\
(s, e_m) & & \\
\Big\downarrow \boxed{\begin{array}{c}\text{iteration of}\\ \tau_{g,i} \text{ at right}\end{array}} & \leftrightarrow & \text{multiplication by } U_i \text{ at right.} \\
(s, t) \in P(2N, m) & &
\end{array}$$

So $(s, t)$ is associated with an operator of $T_{2N}(Q)$ which is called $\mathcal{T}'(s, t)$. Then with these definitions, we can show that $\{\mathcal{T}'(s, t)/(s, t) \in P(2N)\}$ is a basis of $T_{2N}(Q)$.

(i) We can first relate the ideals generated by $\mathcal{T}(e_m, e_m)$ and by $\mathcal{T}'(e_m, e_m)$: writing $E_m$ as a linear combination of the identity and of products of $U_i$ with $i \in \{1, \ldots, m-2\}$, it follows that: the ideal generated by $\mathcal{T}(e_m, e_m)$ is the sum of the ideal generated by $\mathcal{T}'(e_m, e_m)$ and of the ideal generated by an operator which belongs to the ideal generated by $\mathcal{T}'(e_{m-2}, e_{m-2})$.

(ii) This relation allows us to lead a recursion on $m$ to prove that all the words $\mathcal{T}'(s, t)$ are linearly independent by using the results of independence obtained on the $\mathcal{T}(s, t)$. The recursion stops at $m = 2N + 1$ where $\mathcal{T}'(e_m, e_m) = I_d$. Thus the algebra $T_{2N}(Q)$ is indeed the ideal generated by $\mathcal{T}'(e_m, e_m)$, that is $\{\mathcal{T}'(s, t)/(s, t) \in P(2N, 2N + 1)\}$.

The independence arguments ensure that $\mathcal{T}'(P(2N))$ is a basis of $T_{2N}(Q)$. But we also have a direct isomorphism between $\mathcal{T}(P(2N))$ and $\mathcal{T}'(P(2N))$ (by construction) and then we can deduce that $\mathcal{T}(P(2N))$ is a basis of $T_{2N}(Q)$.

The decomposition of $T_{2N}(Q)$ into irreducible components is then

$$T_{2N}(Q) = \bigoplus_m \text{vect.sp.}(\mathcal{T}(P(2N, m))).$$

### 2.3. *Completeness of the 6-vertex and Potts representations*

We have written $T_{2N}(Q) = \bigoplus_m \text{vect.sp.}(\mathcal{T}(P(2N, m)))$. So we see that each irreducible component of $T_{2N}(Q)$ is indeed associated with an $m$ such that $m$ is odd and $m \leqslant 2N + 1$, because all irreducible components have as a basis the elements of $P(2N, m)$ respectively, that is the walks in the Bratelli diagram which arrive at $m$.

We then deduce that the largest number of irreducible two-by-two inequivalent components of a representation of $T_{2N}(Q)$ is, in fact, $N + 1$.

*2.3.1. 6-vertex representation.* As shown by Baxter [5] the ice rule implies that the $U_i$'s relate only states with the same number $n$ of up arrows in a line. There is an associated irreducible component for each $n$ but the components associated with $n = N + k$ and $n = N - k$ are interlaced with the up/down reversion of the spins. So, the irreducible components are indexed by $n = 0, \ldots, N$. Thus the number of irreducible two-by-two inequivalent components is the same as the regular representation one, i.e. $N + 1$.

Hence this representation is complete.

*2.3.2. Potts representation.* The dimension of the state space for a chain of $N$ Potts sites is $Q^N$. From the latter decomposition of $T_{2N}(Q)$ into irreducible components (described in 2.2), this dimension is the sum of dimensions associated with all components.

We have seen that the dimension of the component associated with $m$ in $T_{2N}(Q)$ is the number of walks in the Bratelli diagram which end at $m$. This dimension is then $\mathrm{Card}[S(2N, m)]$ which can be calculated by combinatorial arguments [1].

Jones [6] has shown that the algebra $T_{2N}(Q)$ can be obtained through a certain projection from $T_{2N+2}(Q)$. The properties of this projection and those of our irreducible components allow us to conclude that the degeneracies associated with the $m$th component in the Potts representation of $T_{2N+2}(Q)$ (with $m \in \{3, \ldots, 2N + 3\}$) are the same as those associated with the $(m - 2)$th component in the representation of $T_{2N}(Q)$. Thus these degeneracies depend on $m$ and $N$ only through $p = \frac{1}{2}(2N + 1 - m)$. We denote them $d_p$.

The equality between the dimensions of the state spaces is then

$$Q^N = \sum_{\substack{m=1 \\ m \text{ odd}}}^{2N+1} d_{(2N+1-m)/2} \times \mathrm{Card}[S(2N, m)].$$

We now lead a recursion on $N$ to determine $d_N$ when we know $d_i$ for $i < N$ (we begin from $d_0 = 1$). We find that for $Q > 4$, all the $d_p$s are positive, so at each step of the recursion there is one new component which is, in fact, present in the Potts representation.

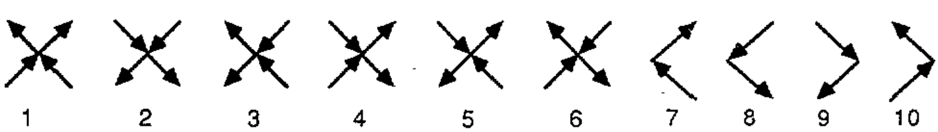
We conclude that each of the $N + 1$ irreducible components is the Potts representation, which is thus complete†.

*2.3.3. The spectra of the transfer matrices.* The transfer matrix is a polynomial of the $U_i$s. The two representations are complete and we thus conclude that:

The spectra of the 6-vertex and Potts transfer matrices are identical (with different degeneracies).

## 3. Bethe ansatz

Let us consider a 6-vertex model on a rotated $M \times N$ lattice (with $2M \times N$ sites), with weights $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$, $\omega_5 = \omega_6 = c$ for the six internal vertices and with weights $\omega_7 = 1$, $\omega_8 = d$, $\omega_9 = e$, $\omega_{10} = 1$ for the four boundary vertices, with $de = 1$ and $d + e = c$:




† This was suggested by V Pasquier.

The partition function is given by

$$Z = \sum_{\text{configurations}} \prod_{\text{vertices}} (\text{weights}) = \mathrm{Tr}((T_1 T_2)^M). \tag{3.1}$$

We have $Z_{\text{Potts}} = Q^{(M\times N)/2} Z_{6V}$ since $x = 1$ at the transition point (see 2.1). Thus

$$-\frac{\mathcal{F}_{\text{Potts}}}{kT} = \frac{\ln Z_{\text{Potts}}}{(\text{Number of Potts sites})} = \frac{1}{2}\ln Q + \frac{\ln Z_{6V}}{(N \times M)}. \tag{3.2}$$

There appear two types of rows of vertices and edges, as explained in section 2.

Let us identify a state of type-1 (or type-2) row of edges by defining the positions of the down arrows of the configuration $1 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant 2N$.

Let $x = \{x_1, \ldots, x_n\}$ and $G(x)$ and $F(x)$ be the element of a vector, defined on a type-1 and type-2 row of edges, respectively, having the property

$$\Lambda F(x) = \sum_y T_1(x, y) G(y) \tag{3.3a}$$

$$\Lambda G(x) = \sum_y T_2(x, y) F(y). \tag{3.3b}$$

We deduce

$$\Lambda^2 G = (T_2 T_1) G \tag{3.4a}$$

and

$$\Lambda^2 F = (T_1 T_2) F. \tag{3.4b}$$

Thus $G$ (respectively $F$) is the eigenvector of the two-row transfer matrix $T_2 T_1$ (respectively $T_1 T_2$) and $\Lambda^2$ is its eigenvalue. It follows that the partition function is asymptotically equal to

$$Z_{6V} \sim \Lambda_{\max}^{2M} \tag{3.5}$$

where $\Lambda_{\max}^2$ is the largest eigenvalue of $(T_1 T_2)$. The correlation length is given by

$$\xi = 1/\ln(\Lambda_{\max}^2/\Lambda_2^2). \tag{3.6}$$

We introduce the Bethe ansatz:

$$F(x_1, \ldots, x_n) = \sum_p \varepsilon_p A(k_1, \ldots, k_n) f(x_1, k_1) \ldots f(x_n, k_n) \tag{3.7a}$$

$$G(x_1, \ldots, x_n) = \sum_p \varepsilon_p A(k_1, \ldots, k_n) f(x_1, k_1) \ldots f(x_n, k_n) \tag{3.7b}$$

and

$$\Lambda = \lambda(k_1) \cdots \lambda(k_n) \tag{3.7c}$$

where the sum extends over all permutations and negations of the $k$ and $\varepsilon_p$ changes sign over all 'mutations'. The quantities $f(x_i, k_i)$ and $g(x_i, k_i)$ are the 'single-particle wave-functions' defined as

$$f(x, k_j) = A_p(k_j)e^{ik_j x} \qquad \text{for } x \text{ even} \tag{3.8a}$$

$$f(x, k_j) = A_i(k_j)e^{ik_j x} \qquad \text{for } x \text{ odd} \tag{3.8b}$$

$$g(x, k_j) = B_p(k_j)e^{ik_j x} \qquad \text{for } x \text{ even} \tag{3.8c}$$

$$g(x, k_j) = B_i(k_j)e^{ik_j x} \qquad \text{for } x \text{ odd.} \tag{3.8d}$$

The Bethe ansatz (3.7) trivially satisfies the general equations (3.3). We define

$$r(k) = \frac{A_p(k)}{A_i(k)}e^{ik} \tag{3.9a}$$

and

$$t(k) = \frac{A_i(k)}{A_p(k)}e^{ik} \tag{3.9b}$$

which verify

$$r(k)t(k) = e^{2ik} \tag{3.10a}$$

$$t(k) = (1 + c \cdot r(k))/(c + r(k)). \tag{3.10b}$$

Setting

$$\delta = (c^2 - 1)/d - c \tag{3.11a}$$

$$\delta' = (c^2 - 1)/e - c \tag{3.11b}$$

which trivially verify

$$\delta\delta' = 1 \tag{3.11c}$$

and

$$\alpha(k) = 1 + \delta t(k) \tag{3.12a}$$

$$\beta(k) = (r(k)t(k))^N (t(k) + \delta') \tag{3.12b}$$

the free boundary conditions give

$$\alpha(k_1)A(k_1, \ldots, k_n) - \alpha(-k_1)A(-k_1, \ldots, k_n) = 0 \tag{3.13a}$$

$$\beta(k_n)A(k_1, \ldots, k_n) - \beta(-k_n)A(k_1, \ldots, -k_n) = 0. \tag{3.13b}$$

We also have the meeting conditions, which give

$$s(k_j, k_{j+1})A(\ldots k_j, k_{j+1} \ldots) - s(k_{j+1}, k_j)A(\ldots k_{j+1}, k_j \ldots) = 0 \tag{3.14}$$

where

$$s(k,k') = 1 + \frac{r(k)}{c} + \frac{c^2-1}{c} r(k') + r(k)r(k'). \tag{3.15}$$

Other equations are found from these by appropriate negations and permutations. We now define

$$B(k,k') = s(k,k') \cdot s(k',-k). \tag{3.16}$$

Hence to obtain the compatibility condition following a scheme given by Owczarek and Baxter [2]

$$\frac{\alpha(-k_j)\beta(k_j)}{\alpha(k_j)\beta(-k_j)} = \prod_{\substack{\ell=1\\ \neq j}}^{n} \frac{B(-k_j,k_\ell)}{B(k_j,k_\ell)} \qquad \forall j \in \{1,\ldots,n\}. \tag{3.17}$$

A solution of the Bethe ansatz equations is given by

$$A(k_1,\ldots,k_n) = \prod_{j=1}^{n} \beta(-k_j) \prod_{1\leqslant j<\ell\leqslant n} \frac{B(-k_j,k_\ell)}{r(k_\ell)}. \tag{3.18}$$

Periodic boundary conditions would give another compatibility equation:

$$e^{2ik_jN} = \prod_{\substack{\ell=1\\ \neq j}}^{n-1} \frac{s(k_\ell,k_j)}{s(k_j,k_\ell)} \tag{3.19}$$

and a solution is given by

$$A(k_1,\ldots,k_n) = s(k_2,k_1)s(k_3,k_2)\cdots s(k_n,k_{n-1}). \tag{3.20}$$

## 4. Parametrization (for free boundaries)

We remember that

$$c = \sqrt{2+Q^{1/2}}. \tag{4.1}$$

We try to express $\xi$ for $Q > 4$, so $c > 2$.

A good parametrization is

$$a = 1 \qquad b = 1 \qquad c = \sinh(2v)/\sinh(v). \tag{4.2}$$

Since $c > 2$, we have $v > 0$, and $Q \longrightarrow 4^+ \Longleftrightarrow v \longrightarrow 0^+$. Setting

$$r(k_j) = \frac{\sinh(\frac{1}{2}v + i\alpha_j)}{\sinh(\frac{1}{2}v - i\alpha_j)} \tag{4.3}$$

this new parametrization gives, using (3.15),

$$\frac{s(k_j, k_\ell)}{s(k_\ell, k_j)} = \frac{\sinh(2v - \mathrm{i}(\alpha_j - \alpha_\ell))}{\sinh(2v + \mathrm{i}(\alpha_j - \alpha_\ell))}. \tag{4.4}$$

We have shown that $\delta\delta' = 1$ for *free boundary conditions* (see (3.11)). Thus

$$\frac{\alpha(-k_j)\beta(k_j)}{\alpha(k_j)\beta(-k_j)} = (\mathrm{e}^{2\mathrm{i}k_j})^{2N}. \tag{4.5}$$

This implies, using (3.16) and (3.10),

$$\left[\frac{\sinh(\frac{1}{2}v + \mathrm{i}\alpha_j)\sinh(\frac{3}{2}v + \mathrm{i}\alpha_j)}{\sinh(\frac{1}{2}v - \mathrm{i}\alpha_j)\sinh(\frac{3}{2}v - \mathrm{i}\alpha_j)}\right]^{2N} = \prod_{\substack{\ell=1\\ \neq j}}^{n} \frac{\sinh(2v + \mathrm{i}(\alpha_j - \alpha_\ell))\sinh(2v + \mathrm{i}(\alpha_j + \alpha_\ell))}{\sinh(2v - \mathrm{i}(\alpha_j - \alpha_\ell))\sinh(2v - \mathrm{i}(\alpha_j + \alpha_\ell))}$$

$$\forall j = 1, \ldots, n \tag{4.6}$$

$\alpha_j$ is real and can be chosen in $[-\pi, \pi]$.

Now consider $r(k_j) = [\sinh(\frac{1}{2}v + \mathrm{i}\alpha_j)]/[\sinh(\frac{1}{2}v - \mathrm{i}\alpha_j)]$. Its modulus is 1. Since

$$\Lambda^2 = \prod_1^n \lambda_j^2 \tag{4.7}$$

and

$$\lambda_j^2 = (cr_j + 1)\left(\frac{c}{r_j} + 1\right) \tag{4.8}$$

the maximum eigenvalue will be obtained if $\arg r_j \simeq 0$. We must thus take a compact set of $\alpha_j$ around zero. We can, moreover, take a symmetric set with respect to zero since (4.6) is invariant under $\alpha_\ell \longrightarrow -\alpha_\ell$. This leads us to take

$$\alpha_{j+1} - \alpha_j \ll 1 \qquad \text{when } N \longrightarrow \infty. \tag{4.9}$$

Let us define $2NR(\alpha)\mathrm{d}\alpha/2\pi$ as the number of $\alpha_j$s between $\alpha$ and $\alpha + \mathrm{d}\alpha$. Setting

$$\phi(\alpha, v) = \mathrm{i}\ln\left(\frac{\sinh(v + \mathrm{i}\alpha)}{\sinh(v - \mathrm{i}\alpha)}\right) \tag{4.10}$$

(4.6) becomes

$$2N(\phi(\alpha_j, \tfrac{1}{2}v) + \phi(\alpha_j, \tfrac{3}{2}v)) = -2\pi I_j + \sum_{\substack{\ell=1\\ \neq j}}^{n}[\phi(\alpha_j - \alpha_\ell, 2v) + \phi(\alpha_j + \alpha_\ell, 2v)]. \tag{4.11}$$

We deduce, in the limit $N \longrightarrow \infty$,

$$\phi'(\alpha, \tfrac{1}{2}v) + \phi'(\alpha, \tfrac{3}{2}v) = -4R(\alpha) + 2\int_{-\pi}^{\pi} \frac{\mathrm{d}\beta}{2\pi} R(\beta)[\phi'(\alpha - \beta, 2v)]. \tag{4.12}$$

We have

$$\phi'(\alpha,\mu)=\frac{2\sinh(2\mu)}{\cosh(2\mathrm{i}\alpha)-\cosh(2\mu)}. \tag{4.13}$$

Developing $R$ in Fourier series

$$R(\alpha)=\sum_{-\infty}^{+\infty}R_m\mathrm{e}^{\mathrm{i}m\alpha} \tag{4.14}$$

we obtain

$$(\phi'(\alpha,\mu))_m=\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{-\mathrm{i}m\alpha}\frac{2\sinh(2\mu)}{\cosh(2\mathrm{i}\alpha)-\cosh(2\mu)}. \tag{4.15}$$

Using a contour integration, we obtain

$$(\phi'(\alpha,\mu))_m=-(1+(-1)^m)\mathrm{e}^{-|m|\mu}. \tag{4.16}$$

The Fourier transform of (4.12) gives

$$(\phi'(\alpha,\tfrac{1}{2}v))_m+(\phi'(\alpha,\tfrac{3}{2}v))_m=-4R_m+(\phi'(\alpha,2v))_m2R_m. \tag{4.17}$$

Since the set is symmetric, $R(\alpha)$ is an even function of $\alpha$, with $R_m=R_{-m}$. Thus

$$R_m=0 \qquad \text{if } m \text{ is odd} \tag{4.18a}$$

$$R_m=\frac{\cosh(|m|\frac{1}{2}v)}{2\cosh(|m|v)} \qquad \text{if } m \text{ is even} \tag{4.18b}$$

$$R_0=\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}R(\alpha)=\frac{\text{number of }\alpha_j}{2N}=\frac{n}{2N}. \tag{4.19}$$

(4.18$b$) gives

$$R_0=\frac{1}{2} \qquad \text{and} \qquad \frac{n}{2N}=\frac{1}{2} \tag{4.20}$$

which could have been predicted, using the symmetry with respect to up/down arrow reversion of the diagonal 6-vertex model.

### 4.1. Free energy

For this calculation, we shall use

$$\Lambda^2=\prod_{j=1}^{n}\lambda^2(k_j) \tag{4.21}$$

(see 3.7). Let us define

$$\psi(\alpha_j)=\lambda^2(k_j). \tag{4.22}$$

Since

$$\lambda^2(k_j) = (cr(k_j) + 1)\left(\frac{c}{r(k_j)} + 1\right) \tag{4.23}$$

we derive

$$\lambda^2(k_j) = \frac{\sinh(\frac{3}{2}v + i\alpha_j)\sinh(\frac{3}{2}v - i\alpha_j)}{\sinh(\frac{1}{2}v + i\alpha_j)\sinh(\frac{1}{2}v - i\alpha_j)}. \tag{4.24}$$

The free energy is given by

$$\frac{1}{2}kT\ln Q + \mathcal{F} = -\frac{kT}{N}\ln\Lambda^2_{\max} = -2kT\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}R(\alpha)\ln\psi(\alpha). \tag{4.25}$$

Developing $\ln(\psi)$ in Fourier series, we note that

$$[\ln\psi]_{-m} = \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}e^{im\alpha}\ln\psi(\alpha). \tag{4.26}$$

Defining

$$\eta_m = \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}e^{im\alpha}\ln\left(\frac{\sinh(\frac{3}{2}v - i\alpha)}{\sinh(\frac{1}{2}v + i\alpha)}\right) = -\int_{-\pi}^{\pi}\frac{d\alpha}{2n}\frac{e^{im\alpha}}{im}\frac{2i\sinh(2v)}{\cosh(v - 2i\alpha) - \cosh(2v)} \tag{4.27}$$

and, using a contour integration, we obtain, for $m < 0$,

$$\eta_m = -\frac{1}{|m|}e^{-(3v/2)|m|}(1 + (-1)^m) \tag{4.28}$$

and, for $m > 0$,

$$\eta_m = \frac{1}{|m|}e^{-|m|v}(1 + (-1)^m). \tag{4.29}$$

In fact we derive that

$$[\ln\psi]_{-m} = \frac{2}{|m|}e^{-|m|v}\sinh(|m|\tfrac{1}{2}v)(1 + (-1)^m). \tag{4.30}$$

We must calculate $[\ln\psi]_0$:

$$[\ln\psi]_0 = \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\ln\frac{\sinh(\frac{3}{2}v - i\alpha)\sinh(\frac{3}{2}v + i\alpha)}{\sinh(\frac{1}{2}v - i\alpha)\sinh(\frac{1}{2}v + i\alpha)}. \tag{4.31}$$

Considering $[\ln\psi]_0$ as a function of $v$: $[\ln\psi]_0 = F(v)$, we derive that

$$\begin{aligned} F'(v) &= \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\left(\frac{3}{2}\frac{2\sinh 3v}{\cosh 3v - \cosh 2i\alpha} - \frac{1}{2}\frac{2\sinh v}{\cosh v - \cosh 2i\alpha}\right) \\ &= -\tfrac{3}{2}(\phi'(\alpha, \tfrac{3}{2}v))_0 + \tfrac{1}{2}(\phi'(\alpha, \tfrac{1}{2}v))_0 = 2. \end{aligned} \tag{4.32}$$

Thus, $R_0[\ln\psi]_0 = \frac{1}{2}\cdot 2v$, and we eventually get

$$\mathcal{F} = -kT\left(\ln\sqrt{Q} + 2v + 2\sum_{m=1}^{\infty}\frac{e^{-2mv}}{m}\tanh(2mv)\right)$$

where

$$2\cosh v = \sqrt{2 + \sqrt{Q}} \tag{4.33}$$

the result of Baxter [5] ch 12-5.

*4.2. Correlation length*

Again we consider our equations for $v \longrightarrow \infty$ [8]: we have $\lambda^2 = \prod_{1 \leqslant i \leqslant n} \psi(\alpha_j)$ with

$$\psi(\alpha_j) = (1 + cr(k_j)) \left(1 + \frac{c}{r(k_j)}\right) = \frac{\sinh(\frac{3}{2}v + \mathrm{i}\alpha_j)\sinh(\frac{3}{2}v - \mathrm{i}\alpha_j)}{\sinh(\frac{1}{2}v + \mathrm{i}\alpha_j)\sinh(\frac{1}{2}v - \mathrm{i}\alpha_j)}. \tag{4.34}$$

An eigenvalue obtained by moving an $I_j$ is degenerate with the largest eigenvalue. The next largest eigenvalue, different from the largest, is obtained by introducing a hole in the last set of $I_j$s.

We note

$$F_1(\alpha_j) = \phi(\alpha_j, \tfrac{1}{2}v) + \phi(\alpha_j, \tfrac{3}{2}v) \qquad \text{and} \qquad F_2(\alpha_j) = \phi(\alpha_j, 2v). \tag{4.35}$$

Let us take the state associated with $I'_j = I_j$ for $j \neq k$ and a hole at the $k$th position (so $n' = N - 1$). Such a hole can be handled as shown by Johnson *et al* [7].

Let us write our compatibility equation for the two distributions of $\alpha_j$s and $\alpha'_j$s, associated, respectively, with $I_j$ and $I'_j$:

$$\begin{aligned} \chi(\alpha)\left(\frac{\mathrm{d}F_1}{\mathrm{d}\alpha} - 2\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha'}{2\pi}R(\alpha')\frac{\mathrm{d}F_2}{\mathrm{d}\alpha}(\alpha-\alpha')\right) \\ = -2\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha'}{2\pi}R(\alpha')\chi(\alpha')\frac{\mathrm{d}F_2}{\mathrm{d}\alpha}(\alpha-\alpha') - (F_2(\alpha-\alpha_k) + F_2(\alpha+\alpha_k)) \end{aligned} \tag{4.36}$$

where $\chi(\alpha)$ is the limit of $2N(\alpha'_j - \alpha_j)$ in the thermodynamic limit. It gives, with the result (4.12),

$$4J(\alpha) = 2\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha'}{2\pi}J(\alpha')\frac{\mathrm{d}F_2}{\mathrm{d}\alpha}(\alpha-\alpha') + (F_2(\alpha-\alpha_k) + F_2(\alpha+\alpha_k)). \tag{4.37}$$

We can solve such an equation through a Fourier series, letting $J_m = \int_{-\pi}^{\pi}\mathrm{d}\alpha\, J(\alpha)\mathrm{e}^{-\mathrm{i}m\alpha}/2\pi$.

We have easily

$$\begin{aligned} &\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{-\mathrm{i}m\alpha}\frac{\mathrm{d}F_2}{\mathrm{d}\alpha}(\alpha) = -(1 + (-1)^m)\mathrm{e}^{-2|m|v} \\ &\int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{-\mathrm{i}m\alpha}(F_2(\alpha-\alpha_k) + F_2(\alpha+\alpha_k)) = 4\frac{\cos(m\alpha_k)}{\mathrm{i}m}((-1)^m - (1 + (-1)^m)\mathrm{e}^{-2|m|v}). \end{aligned} \tag{4.38}$$

So, for $m$ even different from zero:

$$J_m = \frac{\cos(m\alpha_k)}{\mathrm{i}m}\tanh(|m|v). \tag{4.39}$$

We also have

$$\begin{aligned} [(\ln\psi)']_{-m} &= \int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{\mathrm{i}m\alpha}(\ln\psi)'(\alpha) = -2\mathrm{i}(1 + (-1)^m)\mathrm{e}^{-|m|v}\sinh(|m|\tfrac{1}{2}v) \qquad \text{if } m \geqslant 0 \\ &= +2\mathrm{i}(1 + (-1)^m)\mathrm{e}^{-|m|v}\sinh(|m|\tfrac{1}{2}v) \qquad \text{if } m \leqslant 0. \end{aligned} \tag{4.40}$$

The eigenvalue given by the set of $I'_j$ is denoted $\Lambda_2^2$; then, because $\alpha'_j \approx \alpha_j$, we have

$$\ln(\Lambda_2^2) = \sum_1^n \ln(\psi(\alpha'_j)) = \frac{1}{2N} \sum_{1\leqslant j\neq k\leqslant n} 2N(\alpha'_j - \alpha_j)(\ln\psi)'(\alpha_j) - \ln\psi(\alpha_k) + \ln(\Lambda_{\max}^2) \tag{4.41}$$

and hence

$$\begin{aligned}\ln\left(\frac{\Lambda_2^2}{\Lambda_{\max}^2}\right) &= \int_{-\pi}^{\pi} \frac{\mathrm{d}\alpha}{2\pi} J(\alpha)(\ln\psi)'(\alpha) - \ln\Psi(\alpha_k) \\ &= \sum_{-\infty}^{+\infty} J_m[(\ln\psi)']_{-m} - \ln\left(\frac{\sinh(\frac{3}{2}v + \mathrm{i}\alpha_k)\sinh(\frac{3}{2}v - \mathrm{i}\alpha_k)}{\sinh(\frac{1}{2}v + \mathrm{i}\alpha_k)\sinh(\frac{1}{2}v - \mathrm{i}\alpha_k)}\right).\end{aligned} \tag{4.42}$$

Considering $\ln(\Lambda_2^2/\Lambda_{\max}^2)$ as a function of the variable $\alpha_k$ we can easily see that this function has an extremum associated with the greatest $\Lambda_2$ for $\alpha_k = \pi/2$, then

$$-\xi^{-1} = -4\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \mathrm{e}^{-2mv} \sinh(mv)\tanh(2mv) - 2\ln\left(\frac{\cosh\frac{3}{2}v}{\cosh\frac{1}{2}v}\right) \tag{4.43}$$

for our 6-vertex model. Thus

$$\xi = \frac{1}{2\ln(\cosh\frac{3}{2}v/\cosh\frac{1}{2}v) + 4\sum_{m=1}^{\infty}((-1)^m/m)\mathrm{e}^{-2mv}\sinh(mv)\tanh(2mv)} \tag{4.44}$$

with $2\cosh v = \sqrt{2+\sqrt{Q}}$. The behaviour of $\xi$ for $Q \longrightarrow \infty$ is

$$\xi \underset{Q\longrightarrow\infty}{\sim} 2/\ln Q. \tag{4.45}$$

This result can also be obtained using large-$Q$ expansion in the disordered phase. The spin–spin correlation function at distance $r$ is

$$\begin{aligned} c(r) = \left\langle \delta_{\sigma_l\sigma_{l+1}} - \frac{1}{q} \right\rangle &\simeq (\mathrm{e}^{\beta} - 1)^r q^{-r} \\ &= q^{-r/2} \qquad \text{for } \beta \longrightarrow \beta_c. \end{aligned}$$

It is shown in the appendix that $\xi$ is also given by

$$\xi^{-1} = 4\sum_{n=0}^{\infty} \ln\left(\frac{1 + [\sqrt{2}\cosh((\pi^2/2v)(n+\frac{1}{2}))]^{-1}}{1 - [\sqrt{2}\cosh((\pi^2/2v)(n+\frac{1}{2}))]^{-1}}\right) \tag{4.46}$$

with $2\cosh v = \sqrt{2+\sqrt{Q}}$. Its first term gives the analytic form

$$\xi \underset{Q\longrightarrow 4}{\sim} \tfrac{1}{8}\sqrt{2}\exp(\pi^2/\sqrt{Q-4}). \tag{4.47}$$

Johnson *et al* [7] have computed the correlation length of the 6-vertex model near the critical region.

This is also $1/\sqrt{2}$ times the correlation length of the Potts model along the diagonal. The two formulae for this length agree near the critical region as expected from restoration of rotational invariance.

## 5. Conclusion

For the $q = 10$ Potts model, our prediction for the correlation length at the transition point is $\xi = 10.56$. This is larger than estimates from Monte Carlo simulations. In [9] the correlation length at $\beta_t$ is obtained from the spin–spin correlation function projected to non-zero momenta (such a correlation function behaves exponentially with the distance). The quoted result is $\xi = 5.66 \pm 0.09$, it is interpreted as the correlation length in the ordered phase (allowing for the possibility that the correlation length is not the same in the two coexisting phases).

In [10] the correlation length at the peak of the specific heat (at a temperature slightly higher than $\beta_t$) is obtained from the spin–spin correlation function. The correlation function is not projected to a definite momentum, and behaves like an exponential with a power-like prefactor. The quoted result is $\xi = 5.9 \pm 0.7$.

We have shown that our result agrees with the correlation length in the disordered phase, in the large-$q$ limit. A possible interpretation of the discrepancy with the Monte Carlo data is that the correlation length is larger in the disordered phase than in the ordered phase.

## Acknowledgments

We thank A Billoire who proposed this problem and all scientists at the Service de Physique Théorique de Saclay for their cordial welcome. We acknowledge illuminating discussions with V Pasquier, R Balian, M Gaudin, A Morel and J des Cloizeaux.

*Note added in proof.* After completion of this paper we became aware of a calculation of the correlation length of the Potts model in the diagonal direction [11]. Furthermore, formula (4.46) appears implicitly in [12]. We thank Professor Zittartz for pointing out these references.

## Appendix. Behaviour of the correlation length near the limit $Q = 4$

The behaviour of the correlation length near the critical point is known from renormalization group considerations [13]. Our exact result will allow us to rederive this behaviour analytically† by looking for the behaviour of (4.44) near $Q = 4$.

We can rewrite $\xi^{-1}$ by developing this expression in powers of $\mathrm{e}^{-v}$ which gives

$$\frac{\xi^{-1}}{2} = \ln\left(\frac{\cosh\frac{3}{2}v}{\cosh\frac{1}{2}v}\right) - \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}(\mathrm{e}^{mv} - \mathrm{e}^{-mv})\mathrm{e}^{-2mv}(1 - \mathrm{e}^{-4mv})\left(\sum_{k=1}^{\infty}(-1)^{k-1}\mathrm{e}^{-4mv(k-1)}\right) \tag{A.1}$$

$$= \ln\left(\frac{\cosh\frac{3}{2}v}{\cosh\frac{1}{2}v}\right) - \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}(\mathrm{e}^{mv} - \mathrm{e}^{-mv})\sum_{k=1}^{\infty}(-1)^{k-1}\mathrm{e}^{-4kmv}(\mathrm{e}^{2mv} - \mathrm{e}^{-2mv}). \tag{A.2}$$

† This was suggested by R Balian.

This allows us to perform the summation over $m$ according to

$$\frac{\xi^{-1}}{2} = \ln\left(\frac{\cosh\frac{3}{2}v}{\cosh\frac{1}{2}v}\right) + \sum_{k=1}^{\infty}(-1)^k \ln\left(\frac{(1+\mathrm{e}^{-(4k+3)v})(1+\mathrm{e}^{-(4k-3)v})}{(1+\mathrm{e}^{-(4k+1)v})(1+\mathrm{e}^{-(4k-1)v})}\right) \tag{A.3}$$

or

$$\frac{\xi^{-1}}{2} = \frac{1}{2}\sum_{-\infty}^{+\infty}(-1)^k \ln\left(\frac{\cosh((4k+3)\frac{1}{2}v)\cosh((4k-3)\frac{1}{2}v)}{\cosh((4k+1)\frac{1}{2}v)\cosh((4k-1)\frac{1}{2}v)}\right). \tag{A.4}$$

By using the Sommerfeld–Watson method, we replace the summation over $k$ in (A.4) by the contour integral

$$\frac{\xi^{-1}}{2} = \frac{1}{4\mathrm{i}}\oint \frac{\mathrm{d}z}{\sin(\pi z)} \ln\left(\frac{\cosh((4z+3)\frac{1}{2}v)\cosh((4z-3)\frac{1}{2}v)}{\cosh((4z+1)\frac{1}{2}v)\cosh((4z-1)\frac{1}{2}v)}\right) \tag{A.5}$$

over the contour

(A.6)

We integrate by parts (A.5) and find

$$\frac{\xi^{-1}}{2} = 0 - \frac{v}{2\mathrm{i}\pi}\oint \mathrm{d}z\ \ln(\tan(\tfrac{1}{2}\pi z))[\tanh((4z+3)\tfrac{1}{2}v) + \tanh((4z-3)\tfrac{1}{2}v)$$
$$- \tanh((4z+1)\tfrac{1}{2}v) - \tanh((4z-1)\tfrac{1}{2}v)] \tag{A.7}$$

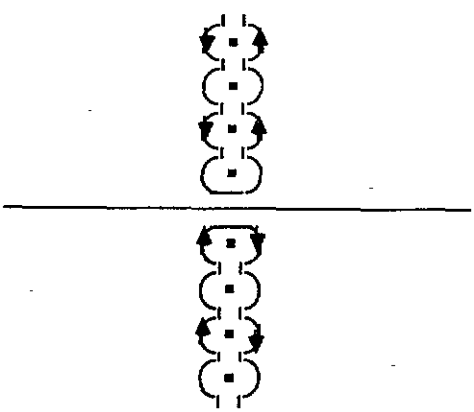
with the appropriate changes of variables we find

$$\frac{\xi^{-1}}{2} = \frac{-2v}{2\mathrm{i}\pi}\oint \mathrm{d}z\ \tanh(2vz)\ln\left(\left(\frac{\mathrm{e}^{\mathrm{i}\pi z+3\mathrm{i}\pi/4}-1}{\mathrm{e}^{\mathrm{i}\pi z+3\mathrm{i}\pi/4}+1}\right)\left(\frac{\mathrm{e}^{\mathrm{i}\pi z-3\mathrm{i}\pi/4}-1}{\mathrm{e}^{\mathrm{i}\pi z-3\mathrm{i}\pi/4}+1}\right)\right.$$
$$\left.\times\left(\frac{\mathrm{e}^{\mathrm{i}\pi z+\mathrm{i}\pi/4}-1}{\mathrm{e}^{\mathrm{i}\pi z+\mathrm{i}\pi/4}+1}\right)\left(\frac{\mathrm{e}^{\mathrm{i}\pi z-\mathrm{i}\pi/4}-1}{\mathrm{e}^{\mathrm{i}\pi z-\mathrm{i}\pi/4}+1}\right)\right) \tag{A.8}$$

i.e.

$$\frac{\xi^{-1}}{2} = \frac{-2v}{2\mathrm{i}\pi}\oint \mathrm{d}z\ \tanh(2vz)\ln\left(\frac{\mathrm{e}^{\mathrm{i}\pi z}+1+\sqrt{2}\mathrm{e}^{\mathrm{i}\pi z}}{\mathrm{e}^{2\mathrm{i}\pi z}+1-\sqrt{2}\mathrm{e}^{\mathrm{i}\pi z}}\right).$$

The poles of $\tanh(2vz)$ are $z = \mathrm{i}(n+\frac{1}{2})\pi/2v$, with residue $-1/2v$. We now can deform the contour (A.6) into



(A.9)

This leads to a new expression for $\xi$:

$$\xi^{-1} = 4\sum_{n=0}^{\infty} \ln\left(\frac{1 + 1/\sqrt{2}\cosh((\pi^2/2v)(n+\frac{1}{2}))}{1 - 1/\sqrt{2}\cosh((\pi^2/2v)(n+\frac{1}{2}))}\right) \tag{A.10}$$

with $2\cosh v = \sqrt{2+\sqrt{Q}}$.

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